Handout 3

1 Two-Country Model with Incomplete Markets

As in the setup we examined previously, we still have a a continuum of households of measure one. The utility maximization problem for a representative household in the home country now takes the form

$$\begin{aligned}
&\max_{[C_t, I_t, K_{t+1}, B_{D,t+1}, B_{F,t+1}]} E_t \sum_{j=0}^{\infty} \beta^j \left(U(C_{t+j}) + V(L_{t+j})\right)) + \\
&+\beta^j \lambda_{t+j} \left[\Pi_{t+j} + w_{t+j} L_{t+j} + R_{kt+j} K_{t+j} + \right.\\
&-P_{Ct+j} C_{t+j} - I_{t+j} - P_{Ft+j} P^*_{Bt+j} B_{Ft+j+1} + P_{Ft+j} B_{Ft+j} \\
&\left. - \int_{s(t+j+1)} \psi_{t+j+1,t+j} B_{Dt+j+1} + B_{D,t+j} \right] + \\
&+\beta^j \gamma_{t+j} \left[(1-\delta) K_{t+j} + I_{t+j} - K_{t+j+1}\right]
\end{aligned}$$

In this setup there are complete financial markets within a country. However, across countries agents only trade a risk-free bond. Our representative agent setup, implies that the only bond to be traded in equilibrium is the foreign risk-free bond.

Notice that this risk-free bond could have been constructed by agents in the complete financial market setting, simply by buying the appropriate quantity of each of the state contingent bonds. Furthermore, do not let yourself be fooled by the "risk-free" name attached to the bond being traded. The availability of the risk-free bond does not make the setup "risk-free". On the contrary, in the incomplete markets case, country specific shocks generate risk that cannot be optimally shared across countries (unlike the case with complete markets).

1.1 The UIP condition

The first-order condition for the risk-free bond $B_{F,t+1}$ from the above maximization problem takes the form

$$\frac{\partial}{\partial B_{Ft+1}} = -\lambda_t P_{Ft} P^*_{B,t} + \beta \int_{s(t+1)|s(t)} \lambda_{t+j+1} P_{Ft+1} Prob(s(t+1)|s(t)) = 0 \tag{1}$$

Using the above first-order condition, define i_t^*, the risk free interest rate in the foreign country at time t, as follows

$$\frac{1}{1+i_t^*} = P_{B,t}^* = \beta \int_{s(t+1)|s(t)} \frac{\lambda_{t+1} P_{Ft+1}}{\lambda_t P_{Ft}} Prob(s(t+1)|s(t)) = \beta E_t \frac{\lambda_{t+1} P_{Ft+1}}{\lambda_t P_{Ft}} \tag{2}$$

Notice that by solving the utility maximization problem for an agent in the foreign country one can also conclude that $\frac{1}{1+i_t^*} = \beta E_t \frac{\lambda_{t+1}^*}{\lambda^* t}$. Which leads to

$$\frac{1}{1+i_t^*} = \beta E_t \frac{\lambda_{t+1} P_{Ft+1}}{\lambda_t P_{Ft}} = \beta E_t \frac{\lambda_{t+1}^*}{\lambda_t^*} \tag{3}$$

Remember that for the complete markets case, we had found that risk was shared across countries so that $\frac{\lambda_{t+1} P_{Ft+1}}{\lambda_t P_{Ft}} = \frac{\lambda_{t+1}^*}{\lambda_t^*}$ for all t. Equation (3) implies that the risk-sharing condition from the complete markets case holds only in expectation under our incomplete market setup. Next, consider the first-order condition for a domestic bond paying off in the state s at time $t+1$:

$$\frac{\partial}{\partial B_{Ds(t+1)}} = \psi_{s(t+1),t} - \beta \frac{\lambda_{s(t+1)}}{\lambda_t} Prob(s(t+1), t) = 0. \tag{4}$$

The risk-free interest rate prevailing in the home country can be found in terms of the price of a domestic bond that would pay a unit of the local currency, regardless of the state of nature. Its price would be $\int_{s(t+1)} \psi_{s(t+1),t}$. Let the risk-free interest rate i_t be defined as

$$\frac{1}{1+i_t} = \int_{s(t+1)} \psi_{s(t+1),t}. \tag{5}$$

Combining equations (4) and (5), one obtains:

$$\frac{1}{1+i_t} = \beta E_t \frac{\lambda_{s(t+1)}}{\lambda_t}. \tag{6}$$

We would like to combine equations (3) and (6) in order to find a relationship between the domestic and foreign risk-free rates. A slight complication is the fact that in equation (3), we have the term $E_t \lambda_{t+1} P_{Ft+1}$. If we could rewrite this term as $E_t \lambda_{t+1} E_t P_{Ft+1}$, then we'd be done. In fact, we are justified in splitting the conditional expectation operator if we are interested in a linear approximation. Then, we can get to this condition:

$$\frac{1}{1+i_t} E_t \frac{P_{Ft+1}}{P_{Ft}} = \frac{1}{1+i_t^*}. \tag{7}$$

Log linearizing, one obtains

$$E_t\tilde{P}_{Ft+1} - \tilde{P}_{Ft} = \tilde{i}_t - \tilde{i}_t^*, \tag{8}$$

from which one can easily see that holders of the foreign risk free bond need to be compensated for an expected appreciation of the home real exchange rate (which implies $E_t\tilde{P}_{Ft+1} < \tilde{P}_{ft}$) by higher foreign real interest rates relative to domestic rates. We'll call this the uncovered interest parity condition.

1.2 The Evolution of Net Foreign Assets

The budget constraint for a household h at time t can be rewritten as:

$$P_{Ft}\frac{1}{1+i_t^*}B_{Ft+1}(h) = P_{Ft}B_{F,t}(h) + \Pi_t(h) + w_tL_t(h) + R_{k,t}K_t(h) \tag{9}$$
$$-P_{Ct}C_t(h) - I_t(h) - \int_s \psi_{t+1,t}B_{Dt+1}(h) + B_{Dt}(h)$$

Summing over households, the domestic borrowing and lending must net out, thus

$$\int_h P_{Ft}\frac{1}{1+i_t^*}B_{F,t+1}(h)dh = \int_h P_{Ft}B_{F,t}(h)dh + \int_h \Pi_t dh \tag{10}$$
$$+\int_h w_tL_t(h) + \int_h R_{k,t}K_t(h)dh - \int_h P_{c,t}C_t(h)dh - \int_h I_t dh$$

Aggregating over all firms f, notice that $\int_f Y_t(f)df$ is the value of domestic output, which can also be written as Y_t. Aggregate profits, Π_t, are given by $\Pi_t = \int_f Y_t(f)df - w_t\int_h L_t(h) - R_{k,t}\int_h K_t(h)dh$. In turn, profits can be rewritten as $\Pi_t = P_tY_t - w_tL_t - R_{k,t}K_t$. Collecting terms:

$$P_{Ft}\frac{1}{1+i_t^*}B_{F,t+1} = P_{Ft}B_{F,t} + Y_t - P_{Ct}C_t - I_t. \tag{11}$$

From the cost minimization problem in the previous handout, remember that $P_{Ct}C_t = C_{Dt} + P_{Ft}C_{Mt}$. Then, substituting in the equation above:

$$P_{Ft}\frac{1}{1+i_t^*}B_{Ft+1} = P_{Ft}B_{F,t} + Y_t - C_{Dt} - P_{Ft}C_{Mt} - I_t. \tag{12}$$

(13)

Also remember that, from the resource constraint, $Y_t - C_{D,t} - I_t = C^*_{Mt}$, therefore:

$$P_{Ft}\frac{1}{1+i^*_t}B_{Ft+1} = P_{Ft}B_{F,t} + C^*_{Mt} - P_{Ft}C_{Mt}. \tag{14}$$

Notice that the terms $C^*_{Mt} - P_{Ft}C_{Mt}$ yield the trade balance. With just one more manipulation we can see that:

$$P_{Ft}B_{Ft+1} - P_{Ft}B_{F,t} = i^*_t\left(P_{Ft}B_{F,t} + C^*_{Mt} - P_{Ft}C_{Mt}\right) + C^*_{Mt} - P_{Ft}C_{Mt}. \tag{15}$$

Notice that we can interpret the sum of the interest payments, $i^*_t\left(P_{Ft}B_{F,t} + C^*_{Mt} - P_{Ft}C_{Mt}\right)$, and the trade balance, $C^*_{Mt} - P_{Ft}C_{Mt}$, as the current account. So, the equation above states that the change in the net-foreign-asset position is equal to the current account.